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Braided Identities, Quantum Groups, and Clifford Algebras[1]

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A Lie algebra in a braided category is constructed within the algebra structure of the positive part of the Drinfeld–Jimbo quantum group of type A_n such that its universal enveloping algebra is a braided Hopf algebra. Similarities with Clifford algebras are discussed.

1. INTRODUCTION

We are interested in the construction of a *quantum Lie algebra* (cf., e.g., Lyubashenko and Sudbery, 1998). The problem is the following: given a Lie algebra $\mathfrak{g}$, we have to construct a deformed Lie algebra $\mathfrak{g}_q$ with a generalized universal enveloping algebra $U(\mathfrak{g}_q)$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{g}_q & \xrightarrow{q\to 1} & \mathfrak{g} \\ \downarrow & & \downarrow \\ U(\mathfrak{g}_q) = U_q(\mathfrak{g}) & \xrightarrow{q\to 1} & U(\mathfrak{g}) \end{array}$$

where $U_q(\mathfrak{g})$ is the Drinfeld–Jimbo quantum group related to $\mathfrak{g}$ and $U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$.

We propose a generalized Lie algebra in the case $\mathfrak{g} = sl_{n+1}^{+}$ (upper triangular matrices) for which there hold a generalized antisymmetry property and a generalized Jacobi identity. Similar structures appear in Gurevich (1986), Woronowicz (1989), Wambst (1993), and Oziewicz *et al.* (1994).

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The algebra and coalgebra structure of the universal enveloping algebra of our deformed Lie algebra are constructed in an analogous way to the classical ones. The antipode map is induced by the existence of the generalized opposite Lie algebra. We obtain a braided Hopf algebra structure on $U_q(\mathfrak{g}) = U(\mathfrak{g}_q)$. Such a braid appear in Lusztig (1993). It is used as an auxiliary tool in order to prove that the quantum Serre relations are compatible with the Hopf algebra coproduct. However, here it is recognized as a part of a braided Hopf algebra.

2. BRAIDED IDENTITIES

If M, N, R are k-modules and $f\colon M \to N$ a linear morphism, we denote $f_1 \equiv f \otimes 1\colon M \otimes R \to N \otimes R$ and $f_2 \equiv 1 \otimes f\colon R \otimes M \to R \otimes N$.

Let A be an associative k-algebra with a multiplication m. Let $\sigma\colon A^{\otimes 2} \to A^{\otimes 2}$ be a linear morphism. Define $f = m - m\sigma\colon A^{\otimes 2} \to A$, f_1, $f_2\colon A^{\otimes 3} \to A^{\otimes 2}$. The following identity holds:

$$m(f_2 - f_1 + f_1\sigma_2 - f_2\sigma_1 + f_2\sigma_1\sigma_2 - f_1\sigma_2\sigma_1) = mm_1(\sigma_1\sigma_2\sigma_1 - \sigma_2\sigma_1\sigma_2) \tag{1}$$

The above identity is an analogy of the Jacobi identity for a bracket f. Such a Jacobi identity will help us to extract a braided Lie algebra from the Drinfeld–Jimbo quantum groups of type A_n.

Lemma 1. Let

$$\sigma f_1 = f_2\sigma_1\sigma_2 \qquad \text{and} \qquad \sigma f_2 = f_1\sigma_2\sigma_1 \tag{2}$$

Then

$$f(f_2\sigma_2 - f_1\sigma_2 + f_1) = mm_1(\sigma_1\sigma_2\sigma_1 - \sigma_2\sigma_1\sigma_2) + mf_1(1 - \sigma_2^2) \tag{3}$$

Another braided identity, linear in m, instead of quadratic as in Lemma 1, is given by Oziewicz and Różański (1994, p. 1084, Corollary 2.6).

There are more versions of (1). For instance, if f satisfies (2), σ is an invertible braid, $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$; then

$$f(f_2 - f_1 + f_1\sigma_2^{-1}) = mf_1(\sigma_2^{-1} - \sigma_2) \tag{4}$$

or if f satisfies (2) just over an element $v \in A^{\otimes 3}$ and σ is a braid, then

$$f(f_2 - f_1)(v) = f_2\sigma_1(v) - f_1\sigma_2(v) \tag{5}$$

Let $A \otimes_\sigma A$ be a k-module $A \otimes A$ with the following product:

$$(a \otimes b)(c \otimes d) = a\sigma(b \otimes c)d \tag{6}$$

Lemma 2. If

$$\sigma m_1 = m_2\sigma_1\sigma_2 \tag{7}$$

$$\sigma m_2 = m_1\sigma_2\sigma_1 \tag{8}$$

then the product (6) is associative. If the unit $1 \in A$ satisfies

$$\forall a \in A, \qquad \sigma(1 \otimes a) = a \otimes 1, \qquad \sigma(a \otimes 1) = 1 \otimes a \tag{9}$$

then $1 \otimes 1$ is the unit for (6).

3. BRAIDS ARISING FROM COMMUTATIVE CONSTRUCTIONS

There are structures built on the flip $x \otimes y \mapsto y \otimes x$, the Clifford algebras and the Drinfeld–Jimbo quantum groups. While the latter is a Hopf algebra, the former has no coalgebra map defined by means of primitives consistent with the algebra structure. However, in both cases, they hide some braid morphisms that enrich the theory. The Clifford algebras are generalized braided Hopf algebras and the positive part of the Drinfeld–Jimbo quantum groups is a genuine universal enveloping algebra of a braided Lie algebra, and such a positive part is a braided Hopf algebra. The braided Hopf algebras were introduced by Majid (1995).

3.1. Clifford Algebra as Braided Quantum Group

Definition 3. Let M be a k-module and f a bilinear form on M. The Clifford algebra $\mathcal{C}\ell(M, f)$ is the k-algebra, with generators the elements of M and relations

$$\forall x, y \in M, \qquad xy + yx = 2f(x, y) \tag{10}$$

Following Đurđevich (1994, p. 151), define $\sigma: M^{\otimes 2} \to M^{\otimes 2}$ by

$$\sigma(x \otimes y) = -y \otimes x - f(x, y)1 \otimes 1$$

$$\sigma(x \otimes 1) = 1 \otimes x, \qquad \sigma(1 \otimes x) = x \otimes 1, \qquad \forall x, y \in M$$

Then σ satisfies the braid equation. We can extend σ to the tensor algebra $M^{\otimes}$ by equations (2) and (9). Then relations (10) hold for such an extension. We induced a linear morphism $\sigma: \mathcal{C}\ell(M, f) \otimes \mathcal{C}\ell(M, f) \to \mathcal{C}\ell(M, f) \otimes \mathcal{C}\ell(M, f)$ which is involutive, and satisfies equations (2) and (9) and the braid equation.

Proposition 4. If f is symmetric, then the map $M \to \mathcal{C}\ell(M, f) \otimes \mathcal{C}\ell(M, f)$, $x \mapsto x \otimes 1 + 1 \otimes x$ can be extended to an algebra morphism

$$\phi: \quad \mathcal{C}\ell(M, f) \to \mathcal{C}\ell(M, f) \otimes_{\sigma} \mathcal{C}\ell(M, f)$$

Đurđevich (1994) developed a counit and an antipode for braided Clifford algebras, therefore, as a particular case for the classical Clifford algebra

$\mathscr{C}\ell(M, f)$. But, because of the lack of multiplicativity of the counit, even in $\mathscr{C}\ell(M, f)$, the obtained structure is not a braided Hopf algebra, but a more general one called a *braided quantum group* (Đurđevich, 1997; see also Đurđevich & Oziewicz, 1996; Oziewicz, 1997; Đurđevich, 2001).

A Clifford algebra $\mathscr{C}\ell(M, f)$ is the universal enveloping algebra of a generalized Abelian Lie algebra; see Example 11. Therefore a braided Jacobi identity (1) for $\mathscr{C}\ell(M, f)$ is a tautology $0 = 0$.

3.2. Drinfeld–Jimbo Quantum Groups of Type A_n Positive

We use the Drinfeld–Jimbo quantum groups in the Lusztig form. The positive part of the Drinfeld–Jimbo quantum group of type A_n is the $\mathbb{C}[q, q^{-1}]$-algebra generated by $E_1, \ldots, E_n$ with relations $E_iE_j^2 - (q + q^{-1})E_jE_iE_j + E_j^2E_i = 0$ if $|i - j| = 1$ and $E_iE_j - E_jE_i = 0$ if $|i - j| > 1$. This algebra is denoted $U_q(sl_{n+1}^+)$. If $q = 1$, $U_q(sl_{n+1}^+)$ collapses to $U(sl_{n+1}^+)$, which is the universal enveloping algebra of sl_{n+1}^+, and $U_q(sl_{n+1}^+)$ is not a Hopf algebra with the usual coproduct of the whole quantum group $U_q(sl_{n+1})$ because the elements $K_i^{\pm 1}$ do not belong to $U_q(sl_{n+1}^+)$. However, later, a *braided* Hopf algebra structure will be constructed on $U_q(sl_{n+1}^+)$.

Define inductively

$$E_{i(i+1)} = E_i, \qquad i = 1, \ldots, n$$

$$E_{i(i+k)} = E_{i(i+k-1)}E(_{i+k-1)(i+k)} - q^{-1}E_{(i+k-1)(i+k)}E_{i(i+k-1)}, \quad 1 < k < n + 1 - i$$

It can be proven that the elements E_{ij}, $1 \le i < j \le n + 1$, generate a braided Lie algebra.

3.3. A Braided Lie Algebra of Type A_3 Positive

Let L_{n+1} be the $\mathbb{C}[q, q^{-1}]$-module with a free basis E_{ij}, $1 \le i < j \le n + 1$. Let $c_{ij,ab} \in \mathbb{Z}$ be such that $[[E_{ij}, E_{ji}], E_{ab}] = c_{ij,ab}E_{ab}$ and let [,] be the usual bracket on sl_{n+1} (Lusztig, 1993, p. 3). Let a total order be given by

$$E_{ij} < E_{ab} \qquad \text{if} \quad (i + j < a + b) \quad \text{or} \quad (i + j = a + b \text{ and } j < b)$$

Define

$$\sigma\colon \quad L_{n+1} \otimes L_{n+1} \to L_{n+1} \otimes L_{n+1}$$

$$\sigma(E_{ij} \otimes E_{ab}) = q^{c_{ij,ab}}E_{ab} \otimes E_{ij}$$

The braid σ is used in Lusztig (1993) as an auxiliary tool in order to prove that the Drinfeld–Jimbo quantum group has a bialgebra structure. In this work, σ is used as a part of a braided Hopf algebra $U_q(sl_{n+1}^+)$.

Define $[\ ,\ ]_q$: $L_{n+1} \otimes L_{n+1} \to L_{n+1}$ by

$$[E_{ij}, E_{ab}]_q = \delta_{ja} E_{ib} - q^{c_{ij,ab}} \delta_{bi} E_{aj}$$

$$\langle E_{ij}, E_{ab} \rangle = \begin{cases} (q - q^{-1}) E_{23} E_{14} & \text{if} \quad E_{ij} = E_{34} \text{ and } E_{ab} = E_{14} \\ 0 & \text{otherwise} \end{cases}$$

Proposition 5. The following relations hold within $U_q(sl_4^+)$:

$$E_{ij}, E_{ab} - q^{c_{ij,ab}} E_{ab} E_{ij} = [E_{ij}, E_{ab}]_q + \langle E_{ij}, E_{ab} \rangle \qquad \text{if} \quad E_{ij} < E_{ab} \tag{11}$$

Proof. Using (5) for $x = E_{12}$, $y = E_{23}$, $z = E_{34}$, and then for $x = E_{13}$, $y = E_{23}$, $z = E_{34}$, etc., we obtain (11) if $E_{ij} < E_{ab}$. ■

This proposition can be generalized to any n. In addition, the relations (11) suggest a generalized commutator, while (4) suggests a Jacobi identity with an involutive switch.

4. GENERALIZED LIE ALGEBRAS AND GENERALIZED UNIVERSAL ENVELOPING ALGEBRAS

Definition 6. A k-module L together with bracket $[\ ,\ ]$: $L \otimes L \to L$, a pseudobracket $\langle\ ,\ \rangle$: $L \otimes L \to L \otimes L$, and presymmetry S: $L \otimes L \to L \otimes L$ is said to be a T-Lie algebra if the following conditions hold:

1. $S^2 = 1$
2. $[\ ,\ ]S = -[\ ,\ ]$, $\langle\ ,\ \rangle S = -\langle\ ,\ \rangle$, $[\ ,\ ]\langle\ ,\ \rangle = 0$
3. $[\ ,\ ]([\ ,\ ]_2 - [\ ,\ ]_1 + [\ ,\ ]_1 S_2) = 0$
4. If $u \in L^{3\otimes}$ such that $S_1(u) \neq u$, $S_2(u) \neq u$ and $S_2 S_1(u) \neq S_2(u)$, then $S[\ ,]_1(u) = [\ ,\ ]_2 S_1 S_2(u)$ and $S[\ ,\ ]_2(u) = [\ ,\ ]_1 S_2 S_1(u)$.

Example 7. Let S: L_{n+1}: $\otimes$ $L_{n+1} \to L_{n+1} \otimes L_{n+1}$ be an involutive map defined by

$$S(E_{ij} \otimes E_{ab}) = q^{c_{ij,ab}} \otimes E_{ij} \qquad \text{if} \quad E_{ij} < E_{ab}$$

and let $\langle\ ,\ \rangle$: $L_{n+1} \otimes L_{n+1} \to L_{n+1} \otimes L_{n+1}$ be a k-module morphism defined by, if $E_{ij} < E_{ab}$,

$$\langle E_{ij}, E_{ab} \rangle = \begin{cases} (q - q^{-1}) E_{aj} \otimes E_{ib} & \text{if} \quad i < a < j \quad \text{and} \quad a < j < b \\ 0 & \text{otherwise} \end{cases}$$

and $\langle\ ,\ \rangle S = -\langle\ ,\ \rangle$. Then L_{n+1} together with $[\ ,\ ]_q \langle\ ,\ \rangle$, and S is a T-Lie algebra.

Example 8. The quantum Lie algebras of Wambst (1993) with involutive symmetry are T-Lie algebras.

Definition 9. Let L be a T-Lie algebra. The universal enveloping algebra $U(L)$ of L is the factor k-algebra of the tensor algebra $L^{\otimes}$ by the two-sided ideal generated by

$$x \otimes y - S(x \otimes y) - [x, y] - \langle x, y \rangle, \; x, y \in L \tag{12}$$

Similar generalized Lie algebras and universal enveloping algebras were studied by Wambst (1993). The main difference with our point of view is that in the generalized universal enveloping algebras defined by Wambst, the additional term $\langle \, , \rangle$ in (12) is a bilinear form. By developing such a bilinear form as in Example 11, the generalized universal enveloping algebras of Wambst are a particular case of (12).

Example 10. Bautista (1998) proved that as k-algebras,

$$U(L_{n+1}) \simeq U_q^+(sl_{n+1})$$

Example 11. The Clifford algebras are universal enveloping algebras of T-Lie algebras. Define $\tilde{M} = M \oplus k$, $\langle \, , \rangle: \tilde{M} \otimes \tilde{M} \to \tilde{M} \otimes \tilde{M}$, $\langle x, y \rangle = f(x, y)1 \otimes 1$, $\forall x, y \in \tilde{M}$, where $f(1, 1) = f(x, 1) = f(1, x) = 0$, $\forall x \in M$. Let S: $\tilde{M} \otimes \tilde{M} \to \tilde{M} \otimes \tilde{M}$, $S(x \otimes y) = -y \otimes x$, $S(x \otimes 1) = 1 \otimes x$, $\forall x \in M$, $S^2 = 1$ Then $\tilde{M}$ is a T-Lie algebra with presymmetry S, bracket $[\, ,] = 0$, and pseudobracket $\langle \, , \rangle$, and its universal enveloping algebra is a Clifford algebra,

$$U(\tilde{M}) \simeq \mathcal{C}\ell(M, f)$$

5. QUANTIZED UNIVERSAL ENVELOPING ALGEBRAS AS BRAIDED HOPF ALGEBRAS: THE CASE $\mathbb{A}_n$

The presymmetry of the T-Lie algebra L_{n+1} preserves the defining relations of $U_q(sl_{n+1}^+)$. Therefore, σ can be extended to σ: $U_q(sl_{n+1}^+) \otimes U_q(sl_{n+1}^+) \to U_q(sl_{n+1}^+) \otimes U_q(sl_{n+1}^+)$ satisfying (7) and (8) relative to the multiplication map m of $U_q(sl_{n+1}^+)$.

Proposition 12. The map $\phi(E_{i(i+1)}) = E_{i(i+1)} \otimes 1 + 1 \otimes E_{i(i+1)}$, $1 \leq i \leq n$, induces an algebra morphism $\phi: U_q(sl_{n+1}^+) \to U_q(sl_{n+1}^+) \otimes_\sigma U_q(sl_{n+1}^+)$.

Proof. Direct computations show that the defining relations of $U_q(sl_{n+1}^+)$ are preserved. ■

Since ϕ is coassociative on the generators E_i, $i = 1, \ldots, n$, ϕ is coassociative on each element of $U_q(sl_{n+1}^+)$.

5.1. Counit

The antipode and the counit of ϕ can be constructed in a similar way to the classical Lie algebra case.

The commutative ring k is a T-Lie algebra in the obvious way and the zero morphism $0\colon L_{n+1} \to k$ is a morphism of T-Lie algebras; then $\epsilon = U(0)$: $U_q(sl^+_{n+1}) \to U(k) \simeq k$ is a morphism of associative k-algebras.

Proposition 13. Let $\mathscr{C} = \{E_1, \ldots, E_n\}$. Then

$$\phi(E_{i_1}) \ldots \phi(E_{i_m}) = E_{i_m} \ldots E_{i_m} \otimes 1 + \sum_j u_j \otimes v_j$$

$$= 1 \otimes E_{i_1} \ldots E_{i_m} + \sum_l a_l \otimes b_l$$

where each u_j, v_j, a_l, b_l is a nonempty product of basic elements in $\mathscr{C}$. It follows that

$$(1 \otimes \epsilon)\phi(E_{i_1}) \ldots \phi(E_{i_m}) = E_{i_1} \ldots E_{i_m} = (\epsilon \otimes 1)\phi(E_{i_1}) \ldots \phi(E_{i_m})$$

Since $\mathscr{C}$ is a generator set of $U_q(sl^+_{n+1})$, we get that ϵ is the counit for ϕ.

5.2. Antipode

A fundamental concept in the Delius and Gould (1996) quantum Lie algebra theory is q-conjugation. In this paper, we use a modified version of q-conjugation.

Definition 14. The $\mathbb{C}$-algebra automorphism $\mathbb{C}[q, q^{-1}] \to \mathbb{C}[q, q^{-1}]$, $q \mapsto q^{-1}$ is called q-conjugation.

Let us fix the ring k as $\mathbb{C}[q, q^{-1}]$ and let L be L_{n+1} as a T-Lie algebra.

Definition 15. 1. The opposite T-Lie algebra L^{op} is the $\mathbb{C}[q, q^{-1}]$-module that coincides with L as a set, has module structure induced by change of fields through q-conjugation, and has T-Lie algebra structure given by

$$[\ ,\ ]^{\mathrm{op}} = -[\ ,\ ]_q, \qquad S^{\mathrm{op}} = S^{-1}, \qquad \langle\ ,\ \rangle^{\mathrm{op}} = -\langle\ ,\ \rangle$$

2. Let m be the product of $U(L)$. The opposite algebra $U(L)^{\mathrm{op}}$ is the $\mathbb{C}[q, q^{-1}]$-algebra that coincides with $U(L)$ as a set, has $\mathbb{C}[q, q^{-1}]$-module structure induced by change of fields through q-conjugation

$$\_ \cdot \_\colon \quad \mathbb{C}[q, q^{-1}] \otimes U(L)^{\mathrm{op}} \to U(L)^{\mathrm{op}}$$

and has product given by

$$m^{\mathrm{op}} = m\sigma = \_ * \_\colon \quad U(L)^{\mathrm{op}} \otimes U(L)^{\mathrm{op}} \to U(L)^{\mathrm{op}}$$

3. We denote by $U(L)^c$ a $\mathbb{C}[q, q^{-1}]$-algebra obtained from $U(L)$ by change of fields through q-conjugation:

$$- \cdot -: \quad \mathbb{C}[q, q^{-1}] \otimes U(L)^c \to U(L)^c$$

Proposition 16. There exists a $\mathbb{C}[q, q^{-1}]$-algebra isomorphism

$$\gamma: \quad U(L) \to U(L)^c$$

such that $q \mapsto q^{-1}$, $E_i \mapsto E_i$, $i, = 1, \ldots, n$.

Proof. The relations

$$\gamma(E_i)^2\gamma(E_j) = (q + q^{-1}) \cdot \gamma(E_i)\gamma(E_j)\gamma(E_i) + \gamma(E_j)\gamma(E_i)^2 \qquad \text{if} \quad c_{ij} = -1$$

$$\gamma(E_i)\gamma(E_j) = \gamma(E_j)\gamma(E_i) \qquad \text{if} \quad c_{ij} = 0$$

are the defining relations of $U_q(sl^+_{n+1})$. ■

Definition 17. The set $\mathscr{B} = \{E_{ij} | 1 \leq i < j \leq n + 1\}$ is called a canonical basis of $L_{n+1.}$

Proposition 18. Let $L = L_{n+1}$.
1. The k-algebra $U(L)^{\mathrm{op}}$ is associative and L^{op} is a basic T-Lie algebra.
2. The map $\eta: L \to L^{\mathrm{op}}$, $x \mapsto -x$ is a T-Lie algebra morphism.
3. There exists an isomorphism of k-algebras,

$$U(L^{\mathrm{op}}) \simeq U(L)^{\mathrm{op}}$$

Proof.
1. The braid equation for σ together with conditions (7) and (8) ensure the associativity property of m^{op}.
2. The following diagrams commute:

$$\begin{array}{ccc} L \otimes L & \xrightarrow{[,]_q} & L \\ {\scriptstyle \eta\otimes\eta}\downarrow & & \downarrow{\scriptstyle \eta} \\ L^{\mathrm{op}} \otimes L & \xrightarrow{[,]_q^{\mathrm{op}}} & L^{\mathrm{op}} \end{array} \qquad \begin{array}{ccc} L \otimes L & \xrightarrow{S} & L \otimes L \\ {\scriptstyle \eta\otimes\eta}\downarrow & & \downarrow{\scriptstyle \eta\otimes\eta} \\ L^{\mathrm{op}} \otimes L^{\mathrm{op}} & \xrightarrow{S^{\mathrm{op}}} & L^{\mathrm{op}} \otimes L^{\mathrm{op}} \end{array}$$

$$\begin{array}{ccc} L \otimes L & \xrightarrow{\langle\, , \rangle} & L \otimes L \\ {\scriptstyle \eta\otimes\eta}\downarrow & & \downarrow{\scriptstyle \eta\otimes\eta} \\ L^{\mathrm{op}} \otimes L^{\mathrm{op}} & \xrightarrow{\langle\, , \rangle^{\mathrm{op}}} & L^{\mathrm{op}} \otimes L^{\mathrm{op}} \end{array}$$

3. Put $\sigma(x \otimes y) = p_{xy}y \otimes x$ and $S(x \otimes y) = q_{xy}y \otimes x$ for any x, y elements in the canonical basis $\mathscr{B}$ of L and where p_{xy}, q_{xy}, $\lambda_i \in \mathbb{C}[q, q^{-1}]$. If $x, y \in \mathscr{B}$ and $x < y$, the defining relations of $U(L)$ can be written as

$$\begin{aligned} &-(xy - p_{xy}yx - [x, y] - \langle x, y\rangle) \\ &\quad = p_{xy}yx - p_{xy}^{-1}p_{xy}xy + [x, y] + \langle x, y\rangle \\ &\quad = x * y - p_{xy} \cdot y * x - [x, y]^{\mathrm{op}} - \langle x, y\rangle^{\mathrm{op}} \end{aligned} \tag{13}$$

This means: $U(L)^{\mathrm{op}}$ is the algebra with defining relations (13), which are the defining relations of $U(L^{\mathrm{op}})$. ■

Proposition 19. There exist a morphism of k-algebras

$$\eta: \quad U(L) \to U(L)^{\mathrm{op}}$$

such that $\eta(x) = -x$, $\forall x \in L$.

Consider the *quantum plane* $\mathbb{A}_q^{2|0}$ defined by the ring

$$\mathbb{A}_q^{2|0} = k\langle x, y\rangle / \langle yx - qxy\rangle$$

where $k\langle x, y\rangle$ means an associative algebra freely generated by x, y.

For positive integers $i \leq m$, we define by the equation in $\mathbb{A}_q^{2|0}$ the numbers

$$\binom{m}{i}_q, \qquad (x + y)^m = \sum_{i=0}^{m} \binom{m}{i}_q x^{m-i} y^i$$

Lemma 20:

$$\sum_{i=0}^{m} \binom{m}{i}_{q^2} (-1)^i q^{i(i-1)} = 0$$

Proof. See Jantzen (1996), p. 6, equation (4) and warning 0.4. ■

Lemma 21. If ι: $U(L)^{\mathrm{op}} \to U(L)^c$ is the natural $\mathbb{C}[q, q^{1-1}]$-linear inclusion and $\kappa = \gamma^{-1} \circ \iota \circ \eta$, then:

1.

$$\phi(E_j)^m = \sum_{i=0}^{m} \binom{m}{i}_{q^2} E_j^{m-i} \otimes E_j^i, \qquad 1 \leq j \leq n$$

2.

$$(E_{j1}^{n_1 - i_1} \otimes E_{j1}^{i_1})(E_{j2}^{n_2 - i_2} \otimes E_{j2}^{i_2}) \ldots (E_{ju}^{n_u - i_u} \otimes E_{ju}^{i_u})$$
$$= q^{\sum_{a<b} c_{j_a j_b} i_a (n_{j_a} - i_b)} E_{j1}^{n_1 - i_1} E_{j2}^{n_2 - i_2} \ldots E_{ju}^{n_u - i_u} \otimes E_{j1}^{i_1} E_{j2}^{i_2} \ldots E_{ju}^{i_u}$$

3.

$$(1 \otimes \kappa)(E_{j1}^{n_1 - i_1} E_{j2}^{n_2 - i_2} \ldots E_{ju}^{n_u - i_u} \otimes E_{j1}^{i_1} E_{j2}^{i_2} \ldots E_{ju}^{i_u})$$
$$= q^{\sum_{a<b} c_{j_a j_b} i_a i_b} E_{j1}^{n_1 - i_1} E_{j2}^{n_2 - i_2} \ldots E_{ju}^{n_u - i_u} \otimes \eta(E_{ju}^{i_u}) \ldots \eta(E_{j1}^{i_1})$$

4. For $j = 1, \ldots, n$,

$$\eta(E_j^i) = (-1)^i q^{i(i-1)} E_j^i$$

5. Denote by m the product of $U_q(sl_{n+1}^+)$. If $n_1, \ldots, n_u$ are positive integers, then

$$m(1 \otimes \kappa)\phi(E_{j_1}^{n_1} \ldots E_{j_u}^{n_u}) = 0, \qquad 1 \leq j_1, \ldots, j_u \leq n.$$

Proof. By straightforward computations. By example, define

$$c = \sum_{a<b} c_{j_a j_b} i_a (n_{j_a} - i_b)$$

Then

$$\begin{aligned}
&m(1 \otimes \kappa)\phi(E_{j_1}^{n_1} \ldots E_{j_u}^{n_u}) \\
&= m(1 \otimes \eta)\phi(E_{j_1})^{n_1} \ldots \phi(E_{j_u})^{n_u} \\
&= \sum_{i_1,\ldots,i_u=0}^{n_1,\ldots,n_u} \binom{n_1}{i_1}_{q^2} \cdots \binom{n_u}{i_u}_{q^2} q^c m(E_{j_1}^{n_1-i_1} \cdots E_{j_u}^{n_u-i_u} \otimes \eta(E_{j_1}^{i_1} \cdots E_{j_u}^{i_u})) \\
&= \sum_{i_1,\ldots,i_u=0}^{n_1,\ldots,n_u} \binom{n_1}{i_1}_{q^2} \cdots \binom{n_u}{i_u}_{q^2} q^{\sum_{a<b} c_{j_a j_b} i_a n_{j_b}} E_{j_1}^{n_1-i_1} \cdots E_{j_u}^{n_u-i_u} \eta(E_{j_u}^{i_u}) \cdots \eta(E_{j_1}^{i_1}) \\
&= \sum_{i_1,\ldots,i_{u-1}=0}^{n_1,\ldots,n_{u-1}} \binom{n_1}{i_1}_{q^2} \cdots \binom{n_{u-1}}{i_{u-1}}_{q^2} q^{\sum_{a<b} c_{j_a j_b} i_a n_{j_b}} E_{j_1}^{n_1-i_1} \cdots E_{j_{u-1}}^{n_{u-1}-i_{u-1}} \\
&\quad \sum_{i_u=0}^{n_u} \binom{n_u}{i_u}_{q^2} (-1)^{i_u} q^{i_u(i_u-1)} E_{j_u}^{n_u} \eta(E_{j_{u-1}}^{i_{u-1}}) \ldots \eta(E_{j_1}^{i_1}) = 0 \quad \blacksquare
\end{aligned}$$

Proposition 22. Let ι: $U(L)^{\mathrm{op}} \to U(L)^{\mathrm{c}}$ the natural $\mathbb{C}[q, q^{-1}]$-linear inclusion. Then, the $\mathbb{C}[q, q^{-1}]$-linear morphism

$$\kappa = \gamma^- \circ \iota \circ \eta: \quad U(L) \to U(L)$$

is the antipode for the coproduct ϕ.

Theorem 23. The $\mathbb{C}[q, q^{-1}]$-algebra $U_q(sl_{n+1}^+)$ with coproduct ϕ, counit ϵ, and antipode κ is a braided Hopf algebra.

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